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A product formula for orthogonal polynomials associated with infinite distance-transitive graphs

Michael Voit

Fachbereich Mathematik, Universität Dortmund, D-44221 Dortmund, Germany

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Abstract

The infinite, locally finite distance-transitive graphs form an extension of homogeneous trees and are described by two discrete parameters. The associated orthogonal polynomials may be regarded as spherical functions of certain Gelfand pairs or as characters of some polynomial hypergroups; they are certain Bernstein polynomials and admit a discrete nonnegative product formula. In this paper we use the graph-theoretic origin of these polynomials to derive the existence of positive dual continuous product and transfer formulas. The dual product formulas will be computed explicitly.

1. Infinite distance-transitive graphs and orthogonal polynomials

1.1. Infinite distance-transitive graphs and the associated hypergroups

Let Γ be the vertex set of a infinite, locally finite, connected undirected graph, which carries the usual metric d. Assume in addition that Γ is distance-transitive which means that for all $v_1, v_2, v_3, v_4 \in \Gamma$ with $d(v_1, v_3) = d(v_3, v_4)$ there exists an automorphism g of Γ satisfying $g(v_1) = v_3$ and $g(v_2) = v_4$. The graphs of this type were classified by MacPherson [Mp]. To describe these graphs, we fix integers $a, b \geqslant 2$ and denote the complete graph with b vertices by C_b; completeness here means that all vertices of C_b are connected. The graph $\Gamma(a, b)$ is now defined as the infinite graph such that precisely a copies of the graph C_b are tacked together at each vertex in a tree-like way, i.e., there are no other cycles in $\Gamma(a, b)$ than those in a single copy of C_b. Obviously, $\Gamma(a, b)$ is an infinite, locally finite, distance-transitive graph, and

E-mail address: michael.voit@mathematik.uni-dortmund.de.

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$\Gamma(a,b)$ is a homogeneous tree precisely for $b=2$. By [Mp], all infinite, locally finite distance-transitive graphs appear in this way. Several aspects of harmonic analysis and probability theory on homogeneous trees and related groups were studied by many authors; see [Cr,CKS,Le,Sa]. It is therefore astonishing that the $\Gamma(a,b)$ did not attract much attention from this point of view. On the other hand, the orthogonal polynomials associated with the $\Gamma(a,b)$ have a long history; see [AW,RV,St], and references therein. We mention that the polynomials in this paper appear in [St] as limits of finite orthogonal polynomials associated with generalized n-gons.

We now fix integers $a,b\geqslant 2$ and equip the group $Aut(\Gamma)$ of all automorphisms of $\Gamma := \Gamma(a,b)$ with the topology of pointwise convergence. Then $Aut(\Gamma)$ is a totally disconnected, locally compact group. Let G be a closed subgroup of $Aut(\Gamma)$ that still acts on Γ in a distance-transitive way. The stabilizer $H\subset G$ of any fixed vertex $e\in\Gamma$ is then a compact open subgroup of G. As G acts transitively on Γ, we may identify the discrete spaces G/H and Γ. The same will be done with the orbit space $\Gamma^H := \{H(v): v\in\Gamma\}$ and the double coset space $G//H := \{HgH : g\in G\}$. Moreover, as Γ is infinite and distance-transitive, we may identify $G//H\simeq\Gamma^H$ with the set $\mathbb{N}_0$ of all nonnegative integers by identifying the orbit $H(v)$ with $d(v,e)\in\mathbb{N}_0$. If $\omega_H\in M^1(G)$ denotes the normalized Haar measure of H, the space

$$M_b(G||H) := \{\mu\in M_b(G) : \omega_H * \mu * \omega_H = \mu\}$$

of all H-biinvariant bounded signed measures on G is a Banach-$*$-subalgebra of $M_b(G)$ with the convolution as product and the total variation norm as norm. $M_b(G||H)$ is isometrically isomorphic with the space $M_b(G//H)\simeq M_b(\mathbb{N}_0)$ of all bounded signed measures on $G//H\simeq\mathbb{N}_0$. Via this isomorphism, $M_b(\mathbb{N}_0)$ receives a canonical Banach-$*$-algebra structure with a convolution which admits almost all properties of a group convolution and which is probability preserving. More precisely, $(\mathbb{N}_0, *)$ is a discrete hypergroup in the sense of C.F. Dunkl, R. Jewett, and R. Spector; for details see the monograph [BH]. The convolution on $M_b(\mathbb{N}_0)$ was computed explicitly in [V1] by counting vertices on Γ and is determined uniquely as the bilinear, weakly continuous extension of the convolution of point measures with

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n} g_{m,n,k}\delta_k \in M^1(\mathbb{N}_0) \tag{1.1}$$

for $m,n\in\mathbb{N}_0$ with

$$g_{m,n,m+n} = \frac{a-1}{a} > 0, \quad g_{m,n,|m-n|} = \frac{1}{a(a-1)^{m\vee n-1}(b-1)^{m\vee n}} > 0,$$

$$g_{m,n,|m-n|+2k+1} = \frac{b-2}{a(a-1)^{m\vee n-k-1}(b-1)^{m\vee n-k}} \geqslant 0$$

for $k=0,\dots,m\wedge n-1$, and, finally, for $k=0,\dots,m\wedge n-2$,

$$g_{m,n,|m-n|+2k+2} = \frac{a-2}{a(a-1)^{m\vee n-k-1}(b-1)^{m\vee n-k-1}} \geqslant 0.$$

The Haar measure on the hypergroup $(\mathbb{N}_0, *)$ is the image of the counting measure on Γ under the canonical projection $p : \Gamma \mapsto \Gamma^H \simeq \mathbb{N}_0$; counting (see [V1]) thus shows that the Haar weights are given by

$$h_0^{(a,b)} := 1, \quad h_n^{(a,b)} = a(a-1)^{n-1}(b-1)^n \quad (n \geqslant 1). \tag{1.2}$$

Using

$$g_{n,1,n+1} = \frac{a-1}{a}, \quad g_{n,1,n} = \frac{b-2}{a(b-1)}, \quad g_{n,1,n-1} = \frac{1}{a(b-1)}$$

we now define a sequence of orthogonal polynomials $(P_n^{(a,b)})_{n\geqslant 0}$ by

$$P_0^{(a,b)} := 1, \quad P_1^{(a,b)}(x) := \frac{2}{a}\sqrt{\frac{a-1}{b-1}}\,x + \frac{b-2}{a(b-1)}$$

and the three-term-recurrence relation

$$P_1^{(a,b)} P_n^{(a,b)} = \frac{1}{a(b-1)} P_{n-1}^{(a,b)} + \frac{b-2}{a(b-1)} P_n^{(a,b)} + \frac{a-1}{a} P_{n+1}^{(a,b)} \quad (n \geqslant 1). \tag{1.3}$$

By induction we then obtain

$$P_m^{(a,b)} P_n^{(a,b)} = \sum_{k=m-n}^{m+n} g_{m,n,k} P_k^{(a,b)} \quad (m, n \geqslant 0). \tag{1.4}$$

Notice that the choice of $P_1^{(a,b)}$ above is in principle arbitrary. Our choice is motivated by the fact that precisely in this case the $P_n^{(a,b)}$ are orthogonal with respect to a measure with support $[-1, 1]$ except for possible singular points; see below. We also notice that for all indices $a, b \in \mathbb{R}$ with $a, b \geqslant 2$, the formulas above remain correct and Eq. (1.1) defines a commutative polynomial hypergroup $K^{(a,b)}$ on $\mathbb{N}_0$. We therefore assume from now on that $a, b \in \mathbb{R}$ with $a, b \geqslant 2$ holds. For details on polynomial hypergroups we refer to [BH,La].

1.2. The orthogonal polynomials

We next discuss some properties of the $P_n^{(a,b)}$. The simple three-term-recurrence (1.3) allows to compute the $P_n^{(a,b)}$ explicitly. In fact, for $z \in \mathbb{C} \backslash \{0, \pm 1\}$ we obtain

$$P_n^{(a,b)}\left(\frac{z + z^{-1}}{2}\right) = \frac{c(z)z^n + c(z^{-1})z^{-n}}{((a-1)(b-1))^{n/2}} \tag{1.5}$$

with

$$c(z) := \frac{(a-1)z - z^{-1} + (b-2)(a-1)^{1/2}(b-1)^{-1/2}}{a(z - z^{-1})}. \tag{1.6}$$

We note that in particular for

$$s_0 := s_0^{(a,b)} := \frac{2-a-b}{2\sqrt{(a-1)(b-1)}}, \quad s_1 := s_1^{(a,b)} := \frac{ab-a-b+2}{2\sqrt{(a-1)(b-1)}} \tag{1.7}$$

we have

$$P_n^{(a,b)}(s_1) = 1, \quad P_n^{(a,b)}(s_0) = (1-b)^{-n} \quad (n \geqslant 0). \tag{1.8}$$

A comparison of (1.5) with the Tchebychev polynomials

$$U_n(\cos t) = \frac{\sin(n+1)t}{\sin t}$$

of the second kind now leads to

$$P_n^{(a,b)}(x) = \frac{a-1}{a((a-1)(b-1))^{n/2}} \times \left(U_n(x) + \frac{b-2}{((a-1)(b-1))^{1/2}} U_{n-1}(x) - \frac{1}{a-1} U_{n-2}(x) \right)$$

for $x \in \mathbb{C}$ (with $U_{-1} = U_{-2} := 0$). The $P_n^{(a,b)}$ thus fit into the Askey–Wilson scheme (see [AW, pp. 26–28]) and are sometimes called Bernstein or Cartier polynomials. The orthogonality relations in [AW] imply that the normalized orthogonality measure $\rho = \rho^{(a,b)} \in M^1(\mathbb{R})$ is

$$d\rho^{(a,b)}(x) = w^{(a,b)}(x)\, dx|_{[-1,1]} \quad \text{for } a \geqslant b \geqslant 2 \tag{1.9}$$

and

$$d\rho^{(a,b)}(x) = w^{(a,b)}(x)\, dx|_{[-1,1]} + \frac{b-a}{b} d\delta_{s_0} \quad \text{for } b > a \geqslant 2 \tag{1.10}$$

with

$$w^{(a,b)}(x) := \frac{a}{2\pi} \frac{(1-x^2)^{1/2}}{(s_1 - x)(x - s_0)}.$$

For $a, b \in \mathbb{R}$ with $a, b \geqslant 2$, the numbers s_0, s_1 satisfy

$$-s_1 \leqslant s_0 \leqslant -1 < 1 \leqslant s_1.$$

Moreover, Eq. (1.5) yields that the dual space

$$\hat{K}^{(a,b)} \simeq \{x \in \mathbb{R} : (P_n^{(a,b)}(x))_{n \geqslant 0} \text{ is bounded}\}$$

of $K^{(a,b)}$ is equal to $[-s_1, s_1]$. This interval obviously contains the support $S := \mathit{supp}\, \rho^{(a,b)}$ of the orthogonality measure.

1.1. Remark. For $p \geqslant 1$ denote the space of L^p-functions on $\mathbb{N}_0$ w.r.t. the Haar measure with weights (1.2) by $L^p(\mathbb{N}_0)$. (1.2) and (1.8) show that for $b > a$, the character α on $K^{(a,b)}$ corresponding to $s_0 \in \mathit{supp}\, \rho$ satisfies $\alpha \in L^p(\mathbb{N}_0)$ precisely for $p > 1 + \ln(a+1)/\ln(b+1)$; in particular, for $b > a$, we have $\alpha \in L^2(\mathbb{N}_0)$.

In the sequel we investigate dual convolutions on S and $\hat{K}^{(a,b)}$. For integers $a, b \geqslant 2$, the existence of a dual probability preserving convolution on S follows from general (hyper)group-theoretic principles; see Theorem 2.4(2) of [V2] or Section 13 of [J]. Such a argument is not available for characters in $\hat{K}^{(a,b)} \backslash S$ and clearly also not for arbitrary $a, b \in \mathbb{R}$, $a, b \geqslant 2$. On the other hand, the existence of a dual probability preserving convolution on $\hat{K}^{(a,b)}$ was established for $b = 2$ and $a \in \mathbb{N}$ with $a \geqslant 2$ (i.e., in the case of homogeneous trees) by Arnaud [A] and Letac [Le] by using positive definite functions on trees. Moreover, for $b = 2$ and $a \in \mathbb{R}$ with $a \geqslant 2$, it was shown in [CKS,Le] that there exists a dual positive convolution on $\hat{K}^{(a,b)} = [-s_1, s_1]$, which is computed explicitly there. We now extend these results of [A,CKS,Le] to the case $b \geqslant 2$. We first restrict our attention to $a, b \in \mathbb{N}$ and the infinite distance-transitive graphs.

2. Existence of positive product and transfer formulas

Let $a, b \in \mathbb{N}$ with $a, b \geqslant 2$, and let $\Gamma = \Gamma(a, b)$ be the associated infinite distance-transitive graph. In the following we sometimes suppress the superscript (a, b). We first investigate which hypergroup characters belong to positive definite functions on the graph. For this we prove the following extension of a result of [A].

2.1. Proposition. *Let $b \geqslant 2$ be an integer, $a \in \mathbb{R}$ with $a \geqslant 2$, and Γ a finite graph formed by N copies of the complete graph C_b in a tree-like way. Then, for all $x \in \mathbb{C}$,*

$$det(P_{d(s,t)}^{(a,b)}(x))_{s,t\in\Gamma} = \frac{2^{Nb}(s_1 - x)^{N(b-1)}(x - s_0)^N}{a^{(b-1)N+1}(b-1)^{N(b/2-1)}(a-1)^{Nb/2}} \prod_{s\in\Gamma} (a - v(s)),$$

where $v(s)$ denotes the number of copies of the graph C_b to which $s \in \Gamma$ belongs.

The proof is based on some known facts about determinants; we include proofs for sake of completeness. Denote the $n \times m$-matrix with all entries equal to 1 by $E_{n,m}$. Moreover, for $c, d \in \mathbb{C}$, let $M_n(c, d)$ be the $n \times n$-matrix with the entry c in the diagonal and d otherwise.

2.2. Lemma. *Let $c, d, e \in \mathbb{C}$ and $m, n \in \mathbb{N}$.*

(1) *$det\, M_n(c, d) = (c - d)^{n-1}(c + (n - 1)d)$.*
(2) *The $mn \times mn$-block matrix*

$$G_{n,m}(c, d, e) := \begin{pmatrix} M_n(c,d) & e \cdot E_{n,n} & e \cdot E_{n,n} & \dots & e \cdot E_{n,n} \\ e \cdot E_{n,n} & M_n(c,d) & e \cdot E_{n,n} & \dots & e \cdot E_{n,n} \\ & & \dots & & \\ e \cdot E_{n,n} & e \cdot E_{n,n} & e \cdot E_{n,n} & \dots & M_n(c,d) \end{pmatrix}$$

satisfies

$$\det G_{n,m}(c,d,e) = (c-d)^{m(n-1)}(c+(n-1)d-ne)^{m-1} \times (c+(n-1)d+n(m-1)e).$$

(3) *If* $c > d$, $\det G_{n,m-1}(c,d,e) > 0$, *and* $\det G_{n,m}(c,d,e) > 0$, *then for* $l = 1, \ldots, n$, *the block matrices*

$$G_{n,m,l}(c,d,e) := \begin{pmatrix} G_{n,m-1}(c,d,e) & e \cdot E_{n(m-1),l} \\ e \cdot E_{l,n(m-1)} & M_l(c,d) \end{pmatrix}$$

also satisfy $\det G_{n,m,l}(c,d,e) > 0$.

Proof. (1) Subtract the first $n-1$ rows $d/((n-2)d+c)$-times from the last one of $M_n(c,d)$. This yields

$$\det M_n(c,d) = \det M_{n-1}(c,d)\left(c - \frac{(n-1)d^2}{(n-2)d+c}\right).$$

Induction now leads to part (1).

(2) Subtract the first $(m-1)n$ rows $\frac{e}{(c+(n-1)d+(m-2)ne)}$-times from the last n rows of $G_{n,m}(c,d,e)$. Then the matrix consisting of the last n rows is given by $(0, 0, \ldots, 0, M_n(\tilde{c}, \tilde{d}))$ with $\tilde{c} := c - h$, $\tilde{d} := d - h$, and

$$h := \frac{e^2(m-1)n}{c+(n-1)d+(m-2)ne}.$$

Therefore,

$$\det G_{n,m}(c,d,e) = \det G_{n,m-1}(c,d,e) \det M_n(\tilde{c}, \tilde{d}).$$

The claim now follows by induction from part (1).

(3) The proof of part (2) also implies that

$$\det G_{n,m,l}(c,d,e) = \det G_{n,m-1}(c,d,e) \det M_l(\tilde{c}, \tilde{d})$$

with

$$\det M_l(\tilde{c}, \tilde{d}) = (c-d)^{l-1}\left[c + (l-1)d - \frac{l(m-1)ne^2}{c+(n-1)d+(m-2)ne}\right].$$

As the $[\ldots]$-term is positive for $l = 0$ and $l = n$ by the assumption and linear in l, the claim follows. $\square$

Proof of Proposition 2.1. We check the determinant formula by induction on the number of copies of the graph C_b which form the graph Γ. We also use the notations of the preceding lemma and suppress the superscripts and the argument in $P_n^{(a,b)}(x)$.

If Γ consists of one copy of C_b, then $(P_{d(s,t)})_{s,t\in\Gamma} = M_b(1, P_1)$, and Lemma 2.2(1) implies

$$det(P_{d(s,t)})_{s,t\in\Gamma} = (1 - P_1)^{b-1}(1 + (b-1)P_1), \tag{2.1}$$

which readily proves the claim for $N = 1$.

Now assume that the determinant formula is correct for some graph Γ formed by a certain positive finite number of copies of the graph C_b. Now consider some vertex $k\in\Gamma$ which belongs to only one subgraph of Γ isomorphic with C_b (note that such a vertex exists!). Denote its neighbors by $k_1, \dots, k_{b-1}$. We now extend Γ at k by $r>0$ new copies of the graph C_b in a tree-like way, and obtain a new graph, say $\tilde{\Gamma}$. The new vertices will be labelled by N_j^i with $1\leqslant i\leqslant r$, $1\leqslant j\leqslant b-1$, i.e., N_j^i is the jth vertex of the ith new graph isomorphic with C_b. The matrix $H := (P_{d(s,t)})_{s,t\in\tilde{\Gamma}}$ then may be written as $H = (H_1, H_2)$ with

$$H_1 = \begin{pmatrix} * & R & P_{d(k,s)}\cdot E_{|\Gamma|-b,1} \\ * & M_{b-1}(1, P_1(x)) & P_1\cdot E_{b-1,1} \\ * & P_1\cdot E_{1,b-1} & 1 \\ * & P_2\cdot E_{b-1,b-1} & P_1\cdot E_{b-1,1} \\ * & P_2\cdot E_{b-1,b-1} & P_1\cdot E_{b-1,1} \\ \cdots & \cdots & \cdots \\ * & P_2\cdot E_{b-1,b-1} & P_1\cdot E_{b-1,1} \end{pmatrix}$$

and where H_2 is given by

$$\begin{pmatrix} P_{d(k,s)+1}\cdot E_{|\Gamma|-b,b-1} & \cdots & \cdots & P_{d(k,s)+1}\cdot E_{|\Gamma|-b,b-1} \\ P_2\cdot E_{b-1,b-1} & \cdots & \cdots & P_2\cdot E_{b-1,b-1} \\ P_1\cdot E_{1,b-1} & \cdots & \cdots & P_1\cdot E_{1,b-1} \\ M_{b-1}(1,P_1) & P_2\cdot E_{b-1,b-1} & \cdots & P_2\cdot E_{b-1,b-1} \\ P_2\cdot E_{b-1,b-1} & M_{b-1}(1,P_1) & \cdots & P_2\cdot E_{b-1,b-1} \\ \cdots & \cdots & \cdots & \cdots \\ P_2\cdot E_{b-1,b-1} & P_2\cdot E_{b-1,b-1} & \cdots & M_{b-1}(1,P_1) \end{pmatrix}.$$

Here $*$ denotes parts of no interest, and all rows of the matrix R contain the entry $P_{d(k,s)}$ exactly $b-2$-times and the entry $P_{d(k,s)-1}$ once; here as well as in the matrices above, s stands for some vertex in Γ with $d(k,s)\geqslant 2$.

Now subtract the column belonging to the vertex k in the H (i.e., the third column of H or H_1) from each of the last $r(b-1)$ columns of H (and H_2, respectively) exactly $\frac{a}{a-1}P_1$-times and add the columns belonging to the vertices $k_1, \dots, k_{q-1}$ (i.e., the columns in the second column of H_1) to these columns exactly $\frac{1}{(a-1)(b-1)}$-times. The three-term-recurrence (1.3) for the P_n ensures that the entries of the modified columns belonging to rows $s\in\Gamma\backslash\{k\}$ are equal to 0. Moreover, the entries of the modified columns belonging to row k also disappear. Using the remaining entries in the modified columns, we conclude that

$$det(P_{d(s,t)})_{s,t\in\tilde{\Gamma}} = det(P_{d(s,t)})_{s,t\in\Gamma}\, det(G_{b-1,r}(c,d,e)) \tag{2.2}$$

with $G_{b-1,r}(c,d,e)$ as in Lemma 2.2 and

$$c := 1 + \frac{1}{a-1}P_2 - \frac{a}{a-1}P_1^2, \quad d := c + P_1 - 1, \quad e := c + P_2 - 1.$$

Therefore, by Lemma 2.2(2),

$$\begin{aligned} det(G_{b-1,r}(c,d,e)) = {} & (1-P_1)^{r(b-2)}((b-2)(P_1-1)-(b-1)(P_2-1))^{r-1} \\ & \times ((b-1)rc + (b-2)(P_1-1) + (b-1)(r-1)(P_2-1)). \end{aligned}$$

A straightforward computation now shows that

$$1 - P_1 = \frac{2\sqrt{a-1}}{a\sqrt{b-1}}(s_1 - x),$$

$$1 + (b-1)P_1 = \frac{2\sqrt{(a-1)(b-1)}}{a}(x - s_0),$$

$$(b-2)(P_1-1) - (b-1)(P_2-1) = \frac{4}{a}(s_1-x)(x-s_0)$$

and

$$\begin{aligned} & (b-1)rc + (b-2)(P_1-1) + (b-1)(r-1)(P_2-1) \\ & \quad = \frac{4(ab-a-b+2)(a-r-1)}{a(a-1)}(s_1-x)(x-s_0). \end{aligned}$$

These formulas, Eq. (2.2) and induction on the number of vertices belonging to at least two copies of the complete graph C_b now imply

$$det(P_{d(s,t)})_{s,t\in\Gamma} = \frac{2^{Nb}(s_1-x)^{N(b-1)}(x-s_0)^N(a-1)^{N/2}}{a^{(b-1)N+1}(b-1)^{N(b/2-1)}(a-1)^{\sum_{s\in\Gamma} v(s)}} \prod_{s\in\Gamma}(a - v(s)).$$

The proposition now follows from $\sum_{s\in\Gamma} v(s) = bN$. $\square$

Proposition 2.1 can be used to decide which characters on the hypergroup $\mathbb{N}_0 \simeq \Gamma(a,b)^H$ may be regarded as a positive definite kernel on $\Gamma(a,b)$ or as a positive definite function on the group G of automorphisms on $\Gamma(a,b)$. For this we recall that a kernel $k : \Gamma(a,b) \times \Gamma(a,b) \to \mathbb{C}$ is called positive definite if for all $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ and $v_1, \ldots, v_n \in \Gamma$ we have

$$\sum_{i,j=1}^{n} c_i \bar{c}_j k(v_i, v_j) \geqslant 0.$$

Proposition 2.1 yields:

2.3. Theorem. *Let $a, b \geqslant 2$ integers, and let G be a locally compact group acting on $\Gamma(a,b)$ in a distance-transitive way. Denote the stabilizer subgroup of a fixed vertex by*

H and identify $G//H$ with $\mathbb{N}_0$ in the obvious way. Then the following statements are equivalent for $x, \tilde{a} \in \mathbb{R}$ with $\tilde{a} \geqslant 2$:

(1) *Either $x \in [s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}]$ and $\tilde{a} \geqslant a$ or $x \in \{s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}\}$ and $\tilde{a} \geqslant 2$;*
(2) *$(v_1, v_2) \mapsto P_{d(v_1,v_2)}^{(\tilde{a},b)}(x)$ is a positive definite kernel on $\Gamma(a,b)$;*
(3) *the mapping $g \mapsto P_{d(gH,e)}^{(\tilde{a},b)}(x)$ is a positive definite function on G.*

Proof. Using the identification $G/H = \Gamma$, we find for all vertices $v_1, v_2 \in \Gamma$ representatives $g_1, g_2 \in G$ with $v_i = g_i H$, and we have

$$d(g_1^{-1} g_2 H, e) = d(g_2 H, g_1(e)) = d(g_2 H, g_1 H).$$

This observation readily implies equivalence (2)$\Leftrightarrow$(3).

In order to check (1) $\Rightarrow$ (2), we first consider $x \in]s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}[$ and $\tilde{a} > a$. Take finitely many arbitrary vertices in $\Gamma(a,b)$, and let T be some finite connected subgraph of $\Gamma(a,b)$ consisting of graphs of type C_b and containing these vertices. Enumerate the elements of T by $s_1, \ldots, s_m$ such that for each $k \in \mathbb{N}_0$ with $b + k(b-1) \leqslant m$, the vertices $s_1, \ldots, s_{b+k(b-1)}$ form a connected subgraph T_k consisting of graphs of type C_b. Then, by Proposition 2.1, $det(P_{d(s_i,s_j)}^{(\tilde{a},b)}(x))_{i,j=1,\ldots,b+k(b-1)} > 0$ for all these k. Moreover, Eq. (2.2) in the proof of Proposition 2.1 and Lemma 2.2(3) imply that

$$det(P_{d(s_i,s_j)}^{(\tilde{a},b)}(x))_{i,j=1,\ldots,r} > 0 \tag{2.3}$$

for all $r = 1, \ldots, m$. This implies part (2) for $x \in]s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}[$ and $\tilde{a} > a$. The case $x \in [s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}]$ and $\tilde{a} \geqslant a$ now follows by taking limits. Moreover, as $P_n^{(\tilde{a},b)}(s_1^{(\tilde{a},b)})$ and $P_n^{(\tilde{a},b)}(s_0^{(\tilde{a},b)})$ are independent of $\tilde{a}$ by (1.8), part (2) also holds for $x = s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}$ and any $\tilde{a} \geqslant 2$.

Now take $\tilde{a} \geqslant 2$ and $x \in \mathbb{R}$ such that the kernel in (2) is positive definite. This in particular implies that $|P_n^{(\tilde{a},b)}(x)| \leqslant 1$ for all $n \geqslant 0$, i.e., we obtain $x \in [-s_1^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}]$. Moreover, the case $x < s_0^{(\tilde{a},b)}$ can be obviously excluded by using Proposition 2.1 for $N = 1$. For $x \in]s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}[$, the case $\tilde{a} < a$ can be also excluded by Proposition 2.1. This completes the proof. □

2.4. Remark. For $\tilde{a} = a$, Theorem 2.3 may be regarded as a discrete analog of well-known results on positive definite functions on the noncompact symmetric spaces of rank one and the associated Jacobi functions; see [FK, p. 265].

Theorem 2.3, the fact that products of positive definite functions on G are again positive definite, the fact that H-biinvariant positive definite functions on G may be regarded as positive definite functions on the hypergroup $\mathbb{N}_0$, and the Bochner theorem on commutative hypergroups (see [J]) now lead to the following result:

2.5. Corollary. *Let $a, b \geqslant 2$ be integers and $x, y \in [s_0^{(a,b)}, s_1^{(a,b)}]$. Then there exists a unique probability measure $\mu_{x,y} \in M^1([-s_1^{(a,b)}, s_1^{(a,b)}])$ with*

$$P_n^{(a,b)}(x) \cdot P_n^{(a,b)}(y) = \int_{-s_1^{(a,b)}}^{s_1^{(a,b)}} P_n^{(a,b)}(z)\, d\mu_{x,y}(z) \quad \textit{for all } n \in \mathbb{N}_0.$$

The following result follows by the same reasons:

2.6. Corollary. *Let $a, b \geqslant 2$ be integers and $\tilde{a} \in \mathbb{R}$ with $\tilde{a} \geqslant a$. Then for each $x \in [s_0^{(\tilde{a},b)}, s_1^{(\tilde{a},b)}]$ there exists a unique probability measure $\mu_x \in M^1([-s_1^{(a,b)}, s_1^{(a,b)}])$ with*

$$P_n^{(\tilde{a},b)}(x) = \int_{-s_1^{(a,b)}}^{s_1^{(a,b)}} P_n^{(a,b)}(z)\, d\mu_x(z) \quad \textit{for all } n \in \mathbb{N}_0.$$

For $b = 2$, the measures $\mu_{x,y}$ and μ_x above were computed explicitly by Letac [Le]; he even observed that in fact both types of measures exist for all real $\tilde{a} \geqslant a \geqslant 2$ with $b = 2$. In the next section we extend the product formula of [Le] to the case $b \geqslant 2$, $b \in \mathbb{R}$.

3. The explicit product formula

Let $a, b \in \mathbb{R}$ with $a, b \geqslant 2$. As the trivial case $a = b = 2$ (which leads to the Tchebychev polynomials of the first kind) has to be treated separately, we assume from now on $a + b > 4$. We now use the Joukowski transform $z \mapsto x(z) := (z + z^{-1})/2$ whose inverse transform is given by $x \mapsto z_{\pm}(x) := x \pm \sqrt{x^2 - 1}$ where the two numbers $z_{\pm}(x)$ satisfy $z_-(x) \cdot z_+(x) = 1$. In the following, it will be convenient to work both with the x- and the z-variables; we agree that from now on x_i always corresponds to z_i ($i \in \mathbb{N}$) where in all formulas the choice of z_i or z_i^{-1} does not matter by symmetry. As a preparation for the kernel appearing in the product formula, we need the following result which extends Proposition 2.1 of [CKS]; see also [RV]:

3.1. Lemma. *Let $z_1, z_2, z_3 \in \mathbb{C} \backslash \{0\}$ satisfying one of the following two conditions:*

(1) $|z_1^{\varepsilon_1} z_2^{\varepsilon_2} z_3^{\varepsilon_3}| < \sqrt{(a-1)(b-1)}$ *for all* $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$,
(2) $z_3 = -\sqrt{a-1}/\sqrt{b-1}$ *and* $|z_1^{\varepsilon_1} z_2^{\varepsilon_2}| < b - 1$ *for all* $\varepsilon_1, \varepsilon_2 = \pm 1$.

Then for $x_i := (z_i + z_i^{-1})/2$ ($i = 1, 2, 3$), the series

$$\sum_{n=0}^{\infty} h_n^{(a,b)} P_n^{(a,b)}(x_1) P_n^{(a,b)}(x_2) P_n^{(a,b)}(x_3) \tag{3.1}$$

converges and is equal to

$$K^{(a,b)}(x_1, x_2, x_3) := \frac{R^{(a,b)}(x_1, x_2, x_3) \prod_{i=1}^{3} \left(ab - a - b + 2 - 2\sqrt{(a-1)(b-1)}\,x_i\right)}{a^2(a-1)(b-1) \prod_{\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1} \left(\sqrt{(a-1)(b-1)} - z_1^{\varepsilon_1} z_2^{\varepsilon_2} z_3^{\varepsilon_3}\right)} \tag{3.2}$$

with

$$\begin{aligned}R^{(a,b)}(x_1,x_2,x_3) &:= (a-2)\prod_{i=1}^{3}(ab-a-b+2+2\sqrt{(a-1)(b-1)}x_i)\\ &\quad + (b-2)\left(ab-a-b+2+2\sqrt{(a-1)(b-1)}\sum_{i=1}^{3}x_i\right)\\ &\quad \times \left(a^2-a-1+(b-1)(-a^3+4a^2-4a+1)\right.\\ &\quad \left.+2(a-1)^{3/2}(b-1)^{1/2}\sum_{i=1}^{3}x_i\right).\end{aligned}$$

Proof. Assume first that $z_i \neq \pm 1$ for all i. In this case, (1.5) and (1.7) show that (3.1) may be written as a sum of eight (or four) geometric sums which converge under the assumptions of the lemma. In particular, the expression in (3.1) is equal to

$$1+\frac{a}{(a-1)}\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1}\frac{c(z_1^{\varepsilon_1})c(z_2^{\varepsilon_2})c(z_3^{\varepsilon_3})\cdot z_1^{\varepsilon_1}z_2^{\varepsilon_2}z_3^{\varepsilon_3}}{\sqrt{(a-1)(b-1)}-z_1^{\varepsilon_1}z_2^{\varepsilon_2}z_3^{\varepsilon_3}} \tag{3.3}$$

The equality of the expressions in (3.3) and (3.2) finally follows by a tedious, but straightforward computation (which was verified by the author also by using MAPLE).

Assume now that $z_1 = \pm 1$ and $z_2, z_3 \neq \pm 1$, and that z_1, z_2, z_3 satisfy one of the conditions of the lemma. Then for any $z \in \mathbb{C}$ with $|z| = 1$ and $z \neq \pm 1$, the lemma holds for (z, z_1, z_2). As for given z_2, z_3, the eight geometric series above converge uniformly in $z \in \mathbb{T}\setminus\{\pm 1\}$, and as the partial sums as well as the right-hand side of (3.2) are continuous in $z \in \mathbb{T}$, it follows readily that the lemma also holds for $(\pm 1, z_1, z_2)$. The remaining cases follow in the same way. □

3.2. Lemma. *The function $K^{(a,b)}$ defined in Lemma* 3.1 *satisfies*

$$K^{(a,b)}(x_1,x_2,x_3) \geqslant 0$$

for all $x_1, x_2 \in [s_0, s_1]$ and $x_3 \in supp\, \rho^{(a,b)}$ for which $K^{(a,b)}$ is nonsingular.

Proof. We first note that Eq. (3.2) yields

$$K^{(a,b)}(s_0,x_1,x_2) = \frac{(a-1)b(b-2)}{a(b-1)^2}\frac{\prod_{i=1}^{2}((1-\frac{1}{z_i\sqrt{(a-1)(b-1)}})(1-\frac{z_i}{\sqrt{(a-1)(b-1)}}))}{\prod_{\varepsilon_1,\varepsilon_2=\pm 1}(1+\frac{z_1^{\varepsilon_1}z_2^{\varepsilon_2}}{b-1})}. \tag{3.4}$$

In particular, $K^{(a,b)}(s_0,x_1,x_2) \geqslant 0$ holds for $x_1, x_2 \in [s_0, s_1]$. It therefore suffices to check that $K^{(a,b)}(x_1,x_2,x_3) \geqslant 0$ for $x_1, x_2 \in [s_0, s_1]$ and $x_3 \in [-1, 1]$. As $x_3 \in [-1, 1]$ is equivalent to $z_3 \in \mathbb{C}$ with $|z_3| = 1$, it follows readily that the numerator of (3.2) is nonnegative for $x_1, x_2 \in [s_0, s_1]$ and $x_3 \in [-1, 1]$. We therefore have to prove that

$$R^{(a,b)}(x_1, x_2, x_3) \geqslant 0 \quad \text{for } x_1, x_2 \in [s_0, s_1], \ x_3 \in [-1, 1]. \tag{3.5}$$

This is obvious for $b = 2$, but the proof is more involved for $b > 2$. All computations below were checked by hand and in addition by using MAPLE. We assume $b > 2$ from now on and write

$$a = q^2 + 1, \quad b = r^2 + 1 \quad \text{with } q \geqslant 1, \ r > 1.$$

The function R is a polynomial in x_1 of degree 2 with a positive leading coefficient. A short computation shows that the minimum of R depending on x_1 is attained at

$$\begin{aligned} x_{1m} := x_{1m}(x_2, x_3) = & \frac{-1}{4q^3 r(r-1)(r+1)} \\ & \times (2qr(q^2+1)(q^2r^2-1)(x_2+x_3) + 4q^2r^2(q^2-1)x_2x_3 \\ & - q^2 + q^4r^2 - q^2r^2 + q^4r^4 - r^2 + r^4q^2 - q^4 + q^6 r^2). \end{aligned}$$

Inspection shows that x_{1m} is linear in x_2 with a negative leading coefficient for all $x_3 \in [s_0, s_1]$. By symmetry, we conclude that x_{1m}, as a function of $x_2, x_3 \in [s_0, s_1]$, is decreasing in these variables. We now discuss three cases depending on the location of x_{1m}:

Case 1: $x_{1m} \leqslant s_0$: As $K^{(a,b)}(s_0, x_2, x_3) \geqslant 0$ by the considerations above, (3.5) is obvious in this case.

Case 2: $x_{1m} \geqslant s_1$: In this case it suffices to check $R(s_1, x_2, x_3) \geqslant 0$ for $x_2, x_3 \in [s_0, s_1]$ with $x_{1m}(x_2, x_3) \geqslant s_1$. By symmetry, this means to prove $R(x_1, x_2, s_1) \geqslant 0$ for $x_1, x_2 \in [s_0, s_1]$ with $x_{1m}(x_1, x_2) \geqslant s_1$. As the equation $x_{1m}(x, s_1) = s_0$ admits the unique solution

$$x^* := -\frac{(q^2r^2+1)(q^4r^2+q^4+2q^2r^2-2q^2-r^2-1)}{4qr(q^4r^2-1)}$$

and as a straightforward computation yields $x^* \leqslant s_0$, we conclude that for $x_2 \in [s_0, s_1]$ we have $x_2 \geqslant x^*$, and hence, by monotonicity, $x_{1m}(x_2, s_1) \leqslant s_0$. It therefore suffices to check $R(s_0, x_2, s_1) \geqslant 0$ which is a consequence Case 1. This completes the proof of (3.5) in this case.

Case 3: $x_{1m} \in [s_0, s_1]$: In this case we have to prove

$$R(x_{1m}(x_2, x_3), x_2, x_3) \geqslant 0$$

for certain x_2, x_3. To describe the set of all these points, we notice that the linear equation $x_{1m}(s_0, x_3) = s_0$ has the unique solution

$$\hat{x}_3 := \frac{q(r^2+q^2)}{r(q^4+1)}.$$

It can be easily checked that $\hat{x}_3 \in [0, s_1]$ for $q, r \geqslant 1$. As $x_2 \geqslant s_0$, we obtain from the monotonicity of x_{1m} that we may restrict our attention to $x_2 \in [s_0, s_1]$ and $x_3 \in [-1, \min\{1, \hat{x}_3\}]$. To check $R(x_{1m}(x_2, x_3), x_2, x_3) \geqslant 0$ in this case, we observe that $R(x_{1m}(x_2, x_3), x_2, x_3)$ is quadratic in x_2. A straightforward computation shows that the extreme value of this function (depending on x_3) is attained at

$$x_{2e}(x_3) := -\frac{(q^2+1)(q^2r^2-1)(q^2+2qrx_3+r^2)}{2qr(2q^3rx_3-2qrx_3-1-3q^2+3q^2r^2+q^4r^2)}.$$

On the other hand, the unique solution of the equation $x_{1m}(x_2, x_3) = s_0$ in x_2 is given by

$$x_{2n}(x_3) := -\frac{\begin{array}{c}2x_3(q^5r^3+q^3r^3-q^3r-qr)\\ +q^6r^2+q^4r^4-q^4r^2+q^4-r^4q^2+q^2r^2-q^2-r^2\end{array}}{2qr(-q^2+q^4r^2-1+q^2r^2+2q^3rx_3-2qrx_3)}.$$

Hence, $x_{2n}(x_3) - x_{2e}(x_3)$ is given by

$$\frac{2q^2(r^2-1)^2(r^2q+q^3-x_3r-q^4rx_3)}{-q^2+q^4r^2-1+q^2r^2+2q^3rx_3-2qrx_3} \times \frac{1}{-3q^2+q^4r^2-1+3q^2r^2+2q^3rx_3-2qrx_3}.$$

It can be readily checked that the denominator of this expression is positive for $x_3 \geqslant s_0$. Moreover, its numerator is nonnegative iff $x_3 \leqslant \hat{x}_3$ holds. Furthermore, as

$$x_{2e}(x_3) - s_0 = \frac{(r^2-1)(r^2q+q^3-rx_3-q^4rx_3)}{r(2q^3rx_3-2qrx_3-1-3q^2+3q^2r^2+q^4r^2)}$$

can be handled in the same way, we conclude that $x_{2n}(x_3) \geqslant x_{2e}(x_3) \geqslant s_0$ for $x_3 \in [s_0, \hat{x}_3]$, and that we have to prove

$$\tilde{R}(x_3) := R(x_{1m}(x_{2e}(x_3), x_3), x_{2e}(x_3), x_3) \geqslant 0$$

for $x_3 \in [-1, \min\{1, \hat{x}_3\}]$. A straightforward computation shows that

$$\tilde{R}(x_3) = \frac{(q^2-1)(r^2-1)Z(x_3)}{2qrx_3(q^2-1)-1-3q^2+3q^2r^2+q^4r^2} \tag{3.6}$$

with

$$\begin{aligned} Z(x_3) := {} & 8q^5r^3x_3^3 + 4(-q^6r^4+q^4r^4-q^8r^4+q^6r^2-q^2r^2-q^4r^2)x_3^2 \\ & + (-2q^7r^5-4q^5r^3-2q^3r)x_3 + r^6q^4 + 4q^6r^4 - 4q^4r^4 + q^{10}r^4 \\ & - 2r^4q^2 + 4q^8r^4 - 4q^6r^2 + r^2 + 4q^4r^2 - 2q^8r^2 + 4q^2r^2 + q^6. \end{aligned}$$

A further straightforward computation now yields that the cubic polynomial Z with positive leading coefficient has two local extrema on $\mathbb{R}$ where the local minimum is attained at

$$\begin{aligned} m := {} & \frac{1}{6q^3r}(-q^2r^2+q^6r^2+q^4r^2-q^4+4+4q^2 \\ & + (1+2q^2+2q^4-2q^2r^2+12q^6r^2-2q^6r^4+q^4r^4-2q^6 \\ & + 2q^8r^4+2q^{10}r^4+q^8-2q^{10}r^2+q^{12}r^4)^{1/2}). \end{aligned}$$

A straightforward computation shows that $m \geqslant 1$ holds for $q, r \geqslant 1$, i.e., the local minimum m is on the right-hand side of the interval $[-1, 1]$. As $\tilde{R}(s_0) \geqslant 0$ by the considerations above, and as $Z(1) = (q^3 - r)^2 (q^2 r^2 - 1)^2 \geqslant 0$ and hence $\tilde{R}(1) \geqslant 0$, we conclude that $\tilde{R} \geqslant 0$ on $[-1, 1]$. This completes the proof. $\square$

3.3. Remark. For $a = 2$ or $b = 2$, we have $s_0 = -s_1$, i.e., Lemma 3.2 implies that the kernel $K^{(a,b)}$ is nonnegative on $\hat{K}^{(a,b)} \times \hat{K}^{(a,b)} \times supp\ \rho$ (except for possible poles). On the other hand, for $a, b > 2$ it may occur for certain points $x_1 \in [-s_1, s_0[\subset \hat{K}^{(a,b)} \backslash supp\ \rho$ and $x_2, x_3 \in [-1, 1] \subset supp\ \rho$ that $K^{(a,b)}(x_1, x_2, x_3) < 0$ holds. Hence, by the arguments in the proof of Theorem 3.4 below, there exists usually no dual positive convolution on the complete dual space $\hat{K}^{(a,b)} = [-s_1, s_1]$. Here is a concrete example: for $a = 5$ and $b = 17$ we have $-s_1 = -65/16$, $s_0 = -5/4$, and $K^{(5,17)}(x, 0, 0) < 0$ for $x \in [-s_1, -5/2[\neq \emptyset$.

Motivated by Corollary 2.5, we now construct an explicit product formula for the polynomials $P_m = P_m^{(a,b)}$ on $[s_0, s_1]$ for all $a, b \in \mathbb{R}$ with $a, b \geqslant 2$ and $a + b > 4$.

3.4. Theorem. *Let $z_1, z_2 \in \mathbb{C} \backslash \{0\}$ and $x_i := (z_i + z_i^{-1})/2$ for $i = 1, 2$. Then for all $m \in \mathbb{N}_0$ the following product formulas hold*:

(1) *If $|z_1^{\varepsilon_1} z_2^{\varepsilon_2}| < \sqrt{(a-1)(b-1)}$ for all $\varepsilon_1, \varepsilon_2 = \pm 1$, then*

$$P_m(x_1) P_m(x_2) = \int_{supp\ \rho} P_m(x) K(x_1, x_2, x)\, d\rho(x).$$

(2) *If $z_1, z_2 \in \mathbb{R}$ with*

$$|z_1 z_2| > \sqrt{(a-1)(b-1)} > \max\{|z_1^{-1} z_2|, |z_1 z_2^{-1}|, |z_1^{-1} z_2^{-1}|\},$$

then

$$P_m(x_1) \cdot P_m(x_2) = A \cdot P_m\left(\frac{z + z^{-1}}{2}\right) + \int_{supp\ \rho} P_m(x) \cdot K(x_1, x_2, x)\, d\rho(x)$$

with

$$z := \frac{z_1 z_2}{\sqrt{(a-1)(b-1)}} \quad \text{and} \quad A := \frac{c(z_1) c(z_2)}{c(z)}.$$

(3) *Let $z_1, z_2 \in \mathbb{R}$ with $|z_1|, |z_2| > 1$. If either $z_1 z_2 = \sqrt{(a-1)(b-1)}$, or if $a \neq b$ and $z_1 z_2 = -\sqrt{(a-1)(b-1)}$, then the product formula of part* (1) *also holds.*

(4) *For all $x_1 \in \mathbb{C}$, $P_m(s_1) \cdot P_m(x_1) = P_m(x_1)$.*

(5) *If $a > b = 2$, then $P_m(-s_1) \cdot P_m(x_1) = P_m(-x_1)$ for all $x_1 \in \mathbb{C}$.*

(6) *If $b > a = 2$, then $s_0 = -s_1$ and for $P_m(-s_1) \cdot P_m(x_1)$ the product formula of part* (1) *holds.*

In particular, for all $x, y \in [s_0, s_1]$ *there exists a unique probability measure* $\mu_{x,y} \in M^1([s_0, s_1])$ *with*

$$P_m(x) \cdot P_m(y) = \int_{s_0}^{s_1} P_m(z)\, d\mu_{x,y}(z) \quad \textit{for all } m \in \mathbb{N}_0.$$

Proof. We first check that cases (1)–(6) cover all possible product formulas for $x, y \in [s_0, s_1]$. In fact, the only possibilities not covered by the main cases (1) and (2) are the cases

$$|z_1 z_2| = \sqrt{(a-1)(b-1)}, \quad z_1 = \pm\sqrt{(a-1)(b-1)}$$

or

$$z_2 = \pm\sqrt{(a-1)(b-1)}.$$

If $z_1, z_2 \neq \pm\sqrt{(a-1)(b-1)}$, then we automatically land up with case (3) (notice that for $a = b$ we have $s_0 = -1$, i.e., for $x_1, x_2 \in [s_0, s_1]$ the case $z_1 z_2 = -\sqrt{(a-1)(b-1)}$ is automatically excluded). Assume now that $z_1 = \pm\sqrt{(a-1)(b-1)}$ with $z_2 \in \mathbb{T}$ holds (notice that the last case $z_2 = \pm\sqrt{(a-1)(b-1)}$ can be handled in the same way). Then the case with a plus sign is case (4), and as $-s_1 = s_0$ holds precisely for $a = 2$ or $b = 2$, the relevant cases of a minus sign are covered by (5) and (6). The nonnegativity of the product formulas now follows immediately from the nonnegativity of K in the relevant regions; see Lemma 3.2. The fact that the measures there are probabilities finally follows from $P_0 = 1$.

We next check the product formulas. Cases (4) and (5) are clear; we just notice that for $b = 2$ we have $P_m(-x) = (-1)^m P_m(x)$.

To prove part (1), we first notice that the isolated point s_0 in *supp* ρ appears precisely for $b > a \geqslant 2$, and that in this case $b - 1 > \sqrt{(a-1)(b-1)}$ holds, i.e., series (3.1) converges for $z_3 := \sqrt{a-1}/\sqrt{b-1}$ and all z_1, z_2 satisfying the conditions of part (1). We therefore obtain from the proof of Lemma 3.1 that in the setting of part (1) series (3.1) converges uniformly for $z_3 := x + \sqrt{x^2 - 1}$ with $x \in \textit{supp}\, \rho^{(a,b)}$. Multiplication of this series with $P_m^{(a,b)}(x)$, integration w.r.t. $\rho^{(a,b)}$, and the orthogonality of the $P_n^{(a,b)}(x)$ now yield the product formula.

Part (6) follows by the same arguments.

We next consider part (3). In this case the definition of the Lebesgue density $w^{(a,b)}$ and Eq. (3.2) show that

$$w^{(a,b)}(x) \cdot K^{(a,b)}(\tilde{x}_1, \tilde{x}_2, x)$$

is a continuous function in the variables $(\tilde{x}_1, \tilde{x}_2, x)$ for $x \in [-1, 1]$ and $(\tilde{x}_1, \tilde{x}_2)$ in a neighborhood of (x_1, x_2). Therefore, as the arguments for the isolated point in $s_0 \in \textit{supp}\, \rho^{(a,b)}$ above for $b > a$ remain available, and as the product formula was already proved for $(\tilde{x}_1, \tilde{x}_2)$ whose z-values satisfy $|\tilde{z}_1 \tilde{z}_2| < \sqrt{(a-1)(b-1)}$, the dominated convergence theorem ensures that the product formula of part (1) also holds for the x_1, x_2 that belong to z_1, z_2 satisfying the conditions of part (3).

It remains to check part (2). According to the proof of Lemma 3.1, we have

$$K(x_1, x_2, x_3) = 1 + \frac{a}{a-1} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1} \frac{c(z_1^{\varepsilon_1})c(z_2^{\varepsilon_2})c(z_3^{\varepsilon_3}) \cdot z_1^{\varepsilon_1} z_2^{\varepsilon_2} z_3^{\varepsilon_3}}{\sqrt{(a-1)(b-1)} - z_1^{\varepsilon_1} z_2^{\varepsilon_2} z_3^{\varepsilon_3}}$$

whenever the z_i satisfy $z_1^{\varepsilon_1} z_2^{\varepsilon_1} z_3^{\varepsilon_1} \neq \sqrt{(a-1)(b-1)}$ for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$. For $m \in \mathbb{N}_0$ let

$$I_m := \int_{-1}^{1} P_m(x) \cdot K(x_1, x_2, x)\, d\rho(x). \tag{3.7}$$

Using the representation

$$w(x) = \frac{a-1}{2\pi a} \frac{1}{|c(x + i\sqrt{1-x^2})|^2 \cdot (1-x^2)^{1/2}}$$

of the weight function, we see that

$$I_m = \rho([-1,1]) \cdot \delta_{m,0} + \sum_{\varepsilon_1,\varepsilon_2=\pm 1} c(z_1^{\varepsilon_1})c(z_2^{\varepsilon_2}) z_1^{\varepsilon_1} z_2^{\varepsilon_2}$$

$$\times \left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} P_m(\cos t)}{(\sqrt{(a-1)(b-1)} - z_1^{\varepsilon_1} z_2^{\varepsilon_2} e^{it}) c(e^{-it})}\, dt \right], \tag{3.8}$$

where the expressions $[\ldots]$ are equal to

$$H^m_{\varepsilon_1,\varepsilon_2} := \frac{1}{2\pi i} \oint_{|w|=1} R^m_{\varepsilon_1,\varepsilon_2}(w)\, dw$$

with

$$R^m_{\varepsilon_1,\varepsilon_2}(w) := \frac{P_m(\frac{w+w^{-1}}{2})}{(\sqrt{(a-1)(b-1)} - z_1^{\varepsilon_1} z_2^{\varepsilon_2} w) c(w^{-1})}$$

$$= \left(\frac{c(w)w^m + c(w^{-1})m^{-m}}{c(w^{-1})} \right) \frac{1}{(\sqrt{(a-1)(b-1)} - z_1^{\varepsilon_1} z_2^{\varepsilon_2} w)((a-1)(b-1))^{m/2}}.$$

These integrals over the unit circle will be evaluated by using the residue theorem. The integrand $R^m_{\varepsilon_1,\varepsilon_2}$ has the following relevant singularities and residues:

(a) For $m \geqslant 1$, we have the pole 0 with residue

$$Res_0(R^m_{\varepsilon_1,\varepsilon_2}) = \frac{(z_1^{\varepsilon_1} z_2^{\varepsilon_2})^{m-1}}{((a-1)(b-1))^m}.$$

Summation over $\varepsilon_1, \varepsilon_2 = \pm 1$ as in Eq. (3.8) with the coefficients there shows that the pole 0 for $m \geqslant 1$ leads to a summand $P_m(x_1)P_m(x_2)$ in I_m.

(b) A second pole of $R^m_{\varepsilon_1,\varepsilon_2}$ is given by

$$w(\varepsilon_1, \varepsilon_2) := \sqrt{(a-1)(b-1)}/(z_1^{\varepsilon_1} z_2^{\varepsilon_2}).$$

Under the conditions of part (2), this pole is inside the unit circle precisely for $\varepsilon_1 = \varepsilon_2 = 1$; the residue is given in this case by

$$Res_{w(\varepsilon_1,\varepsilon_2)}(R^m_{\varepsilon_1,\varepsilon_2}) = -\frac{P_m((w(\varepsilon_1,\varepsilon_2) + w(\varepsilon_1,\varepsilon_2)^{-1})/2)}{z_1 z_2 c(w(\varepsilon_1,\varepsilon_2)^{-1})}.$$

Therefore, this pole leads to the summand $-AP_m((z+z^{-1})/2)$ (with A and z as in part (2) of the theorem).

(c) A further pole appears for w satisfying $c(w^{-1}) = 0$. This holds precisely for $w = \sqrt{(a-1)(b-1)}$, which is outside the unit circle, and for $w_0 := -\sqrt{(a-1)/(b-1)}$, which is outside the unit circle for $a>b$ and inside for $a<b$ (the case $a=b$ will be considered in the end of the proof). In the first case we have no contribution. Assume now that $a<b$ holds. In this case a straightforward computation shows that

$$Res_{w_0}(R^m_{\varepsilon_1,\varepsilon_2}) = \frac{a(b-a)}{b(a-1)} \frac{(-1)^m (\frac{a-1}{b-1})^{(m-2)/2}}{b-1-z_1^{\varepsilon_1} z_2^{\varepsilon_2}}.$$

Therefore, this pole leads to the summand

$$A_1 := \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{a(b-a)}{b(a-1)} \frac{c(z_1^{\varepsilon_1})c(z_2^{\varepsilon_2}) z_1^{\varepsilon_1} z_2^{\varepsilon_2} (-1)^m (\frac{a-1}{b-1})^{(m-2)/2}}{b-1-z_1^{\varepsilon_1} z_2^{\varepsilon_2}}$$

of I_m. Moreover, a comparison with (3.3) and a straightforward computation show

$$A_1 = -\frac{b-a}{b} \cdot K(s_0, x_1, x_2) P_m(s_0) \quad (b>a).$$

Summarizing, we obtain from (a)–(c) that part (2) of the theorem holds for $b \neq a$ and for z_1, z_2 with

$$\sqrt{(a-1)(b-1)}/(z_1^{\varepsilon_1} z_2^{\varepsilon_2}) \neq -\sqrt{(a-1)/(b-1)}$$

in which case the simple poles in (b) and (c) are equal and a double pole appears. This latter case can be again settled by considering small perturbations of z_1, z_2 and then using the already known results together with the dominated convergence theorem applied to the integral in Eq. (3.7). The same argument also settles the case $a = b$ by considering $a<b$ with $a \to b$. This completes the proof of the theorem. $\square$

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